Nonlinear Distortion in Feedback Systems

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We give a method for determining the distortion effect of a nonlinearity in a feedback loop.

I. INTRODUCTION

Desoer gives an interesting analysis of distortion resulting from a nonlinearity of the form $v + \epsilon v^m$ (m an odd integer) in a feedback loop.[1] Sandberg considers virtually the same problem for nonlinearities with upper and lower bounds on the slope.[2] On page 2546 of his work, Sandberg suggests that Desoer's result may be sharpened. Our purpose is to show how a small modification of Desoer's analysis might give this sharpening and extend its applicability.

Desoer's method is to find conditions for a particular mapping to be a contraction in a ball. The method presented in another work is particularly suited to that problem and will be applied in this paper.[3] The problem of distortion in nonlinear systems is also considered in References 4 and 5 among other papers.

II. NOTATION AND PRELIMINARIES

The feedback loop (with unity feedback for simplicity) is assumed to be described by

$$y = NL(r - y) \tag{1}$$

where the input r and output y are in some Banach space. L and N are linear and nonlinear operators, respectively, mapping the Banach space into itself. We need not, at this point, specify which Banach space we are working in. Rather, we refer the reader to Reference 2 for details on two Banach spaces of interest for analysis of nonlinear feedback loops.* In particular, Reference 2 shows how to evaluate

* It must be verified that the Banach space (or an appropriate subset) is mapped into itself by the nonlinearity. In particular, nonlinearities such as described by polynomials do not map L_2 into itself.

the norm of the linear operator when it is defined by a convolution operation or by a transfer function (frequency response).

III. THE PROBLEM OF DISTORTION

Suppose that

$$N(x) = x + \epsilon P(x). \tag{2}$$

Then the loop is linear if $\epsilon = 0$ and it is of interest to determine how the loop response differs for a nonzero ϵ. This difference is called the distortion. On the other hand, we might consider some fixed $|\epsilon| > 0$ and determine how small the input r must be in order that the distortion is sufficiently small. This latter question assumes that $P(x)$ is of an order less than x as $x \to 0$.

The following manipulation is convenient for this problem. From equations 1 and 2 we have

$$y = L(r - y) + \epsilon P[L(r - y)]. \tag{3}$$

If we assume that $(I+L)$ has a bounded inverse where I is the identity map,* we obtain

$$y = (I + L)^{-1}Lr + (I + L)^{-1}\epsilon P[L(r - y)].$$

Then, if z is the response of the linearized loop,

$$z = (I + L)^{-1}Lr.$$

And if ξ represents the distortion,

$$\xi = y - z,$$

we have

$$\begin{aligned} \xi &= \epsilon(I + L)^{-1}P[z - L\xi] \\ &\equiv M(\xi). \end{aligned}$$

We are thus interested in finding a fixed point of the operation $M(\xi)$. In particular, how large is ξ? To solve this problem, we use a convenient modification of the contraction mapping fixed point theorem.

IV. THE CONTRACTION MAPPING THEOREM

Let X be a complete metric space (witb metric d) containing the closed set Ω and let F map Ω into itself. F is a contraction mapping if there is an $\alpha \in [0, 1)$ such that

$$d[F(x), F(x')] \leqq \alpha d(x, x') \qquad (x, x' \in \Omega).$$

* For conditions for the existence of this bounded inverse, see Reference 2, especially p. 2538.

The contraction mapping theorem (Reference 6, p. 627) states that if F is a contraction mapping then there is a unique $x^* \in \Omega$ such that $x^* = F(x^*)$, that is, x^* is a fixed point of the operation F. Also, x^* is the limit of a sequence $\{x_n\}$ where

$$x_{n+1} = F(x_n)$$

and x_o is any element of Ω.

One aspect of using the above theorem is finding the appropriate set Ω mapped into itself. Often, the contraction mapping theorem is used when Ω is the whole space, that is, F is globally Lipschitzian. The analysis of Reference 1 may be viewed as a method of determined a ball about the origin such that a mapping is a contraction in that ball. The general problem of simultaneously trying to determine a set mapped into itself such that a mapping is contraction on that set is discussed in Reference 3. The following simple theorem from Reference 3 is useful in this direction.

Theorem: Let B be a Banach space. F maps B into itself and $x_0 \in B$. It is assumed that

(i) F has a derivative at all $x \in B$

(ii) There is a nondecreasing function g such that if $x \in B$, then

$$\| F'(x) \| \leqq g(\| x - x_0 \|)$$

(iii) There is an $\alpha \in [0, 1)$ such that

$$g\left(\frac{k}{1 - \alpha}\right) \leqq \alpha$$

where

$$k \geqq \| F(x_0) - x_0 \|.$$

Then there is a unique $x^* \in \Omega$ such that

$$x^* = F(x^*)$$

where

$$\Omega = \left\{x : x \in B,\ \| x - x_0 \| \leqq \frac{k}{1 - \alpha}\right\}.$$

Remarks: See chapter XVII of Reference 6 for a general discussion of differentiation in Banach spaces.

It is often a straightforward matter to find an appropriate function g as we shall see in the distortion problem of this paper.

V. SOLUTION OF THE DISTORTION PROBLEM

To apply the preceding theorem to the distortion problem of Section III, we first find $M'(\xi)$, then a nondecreasing g such that

$$\| M'(\xi) \| \leqq g(\| \xi - \xi_0 \|) = g(\| \xi \|) \qquad (\xi_0 = 0). \tag{4}$$

And with $P(0) = 0$ (for simplicity),

$$\begin{aligned} \| M(\xi_0) - \xi_0 \| &= \| \epsilon(I + L)^{-1}P(z) \| \\ &\leqq k. \end{aligned} \tag{5}$$

We must finally find an $\alpha \, \varepsilon [0.1)$ such that

$$g\left(\frac{k}{1 - \alpha}\right) \leqq \alpha. \tag{6}$$

With $(I+L)^{-1}$ and L both assumed to be bounded linear operators, we have

$$\| M'(\xi) \| = \| \epsilon(I + L)^{-1}P'(z - L\xi)L \| \tag{7}$$

(assuming that P has a Fréchet derivative). It should be clear that our method of analysis is not restricted to nonlinearities described by functions of the form of $v + \epsilon v^m$ as used in Reference 1. For the case of the space of continuous real valued functions with the sup norm* and

$$P(x) = x^m \qquad m \text{ an integer} > 1 \text{ (not necessarily odd)} \tag{8}$$

we have that

$$P'(x) = mx^{m-1}. \tag{9}$$

Then, using equations 7 and 9,

$$\begin{aligned} \| M'(\xi) \| &\leqq | \epsilon | \cdot \| (I + L)^{-1} \| \cdot m \cdot \| (z - L\xi)^{m-1} \| \cdot \| L \| \\ &\leqq m \, | \epsilon | \cdot \| (I + L)^{-1} \| \, (\| z \| + \| L \| \cdot \| \xi \|)^{m-1} \, \| L \| \\ &\equiv g(\| \xi - \xi_0 \|) \\ &= g(\| \xi \|) \qquad (\xi_0 = 0). \end{aligned} \tag{10}$$

Now, using equations 5 and 6, we obtain

$$m \, | \epsilon | \cdot \| (I + L)^{-1} \| \cdot \| L \| \left(\| z \| + \frac{\| L \| \cdot | \epsilon | \cdot \| (I + L)^{-1} \| \cdot \| z \|^m}{1 - \alpha} \right)^{m-1} \leqq \alpha. \tag{11}$$

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* What might be considered to be a disadvantage of using this space is that the norms of the linear operator are expressed in terms of impulse responses rather than frequency responses.

Condition (11) could be used in several ways. For a fixed ϵ, we could determine how small $\| z \|$ has to be in order for there to be an $\alpha \in [0, 1)$ satisfying (11) and thus get a bound on the distortion $\| \xi \|$. The emphasis in Reference 1 is in determining how small ϵ should be with linearized outputs satisfying $\| z \| < 1$ and the distortion $\| \xi \| < \frac{1}{2}$ in order for the method of successive approximations to converge at a given rate ($\alpha = \frac{1}{4}$). The discussion on page 2546 of Sandberg's article[2] assumes the following conditions (in our notation):

$$\begin{aligned} m &= 3 \\ \| L \| &= 100 \qquad\qquad (12) \\ \| (I + L)^{-1} \| &= 2. \end{aligned}$$

Then, (11) becomes

$$600 \mid \epsilon \mid \left(1 + \frac{200 \mid \epsilon \mid}{3/4}\right)^2 \leqq \frac{1}{4}. \qquad (13)$$

If $|\epsilon|$ is less than about 1/2900 (actually a little larger, then (13) is satisfied. Then, the distortion ξ satisfies

$$\begin{aligned} \| \xi \| &\leqq \frac{k}{1 - \alpha} \\ &\leqq \mid \epsilon \mid \tfrac{4}{3} \| (I + L)^{-1} \| \cdot \| z \|^m \\ &\leqq \tfrac{8}{3} \mid \epsilon \mid. \qquad (14) \end{aligned}$$

The bound obtained using equation 25 of Desoer's article[1] is $\mid \epsilon \mid \leqq 1/(2150 \cdot 2900)$, a substantially smaller bound.

VI. CONCLUSION

Notice that since we do not require the mapping to be a contraction in the whole space, we only get uniqueness in Ω, the ball of radius $k/(1-\alpha)$. However, the result may be strengthened by also seeking the largest contraction constant α, satisfying condition *iii* of the theorem. Then the fixed point is also unique in the larger ball. On the other hand, uniqueness information might be available from another source (for example, a property of a differential equation).

We notice that the existence of derivatives in the theorem may actually be relaxed if there is a nondecreasing function suitably bounding Lipschitz constants. We also mention the possibility of using transformations to facilitate the application of the result.

The following may be helpful in visualizing the application of the contraction mapping theorem.* Assume that condition *iii* of the theorem of Section IV is satisfied with equality, that is,

$$g\left(\frac{k}{1-\alpha}\right) = \alpha.$$

The radius of the ball Ω is $k/(1 - \alpha)$. Letting

$$r = \frac{k}{1-\alpha}$$

the condition is seen to be

$$g(r) = 1 - \frac{k}{r}$$

which Fig. 1 shows pictorially.

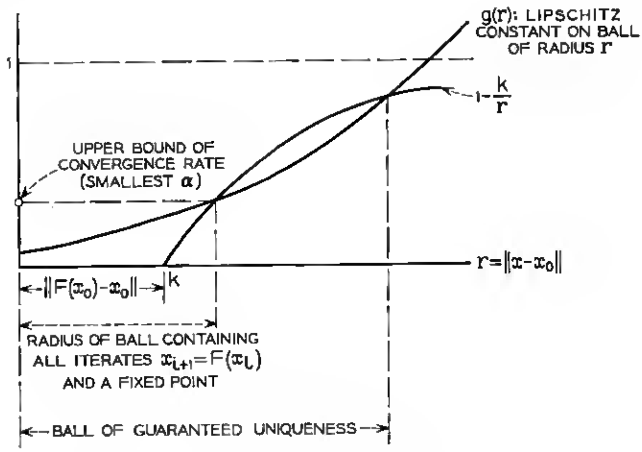

Fig. 1 — Contraction mapping theorem.

VII. ACKNOWLEDGEMENT

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